LIMIT CORRELATION FUNCTIONS FOR FIXED TRACE RANDOM MATRIX ENSEMBLES

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Abstract. Universal limits for the eigenvalue correlation functions in the bulk of the spectrum are shown for a class of non-determinantal random matrices known as the fixed trace or the Hilbert-Schmidt ensemble. These universal limits have been proved before for determinantal Hermitian matrix ensembles and for some special classes of the Wigner random matrices.

1. Introduction and the statement of the result

Let $\mathcal{H}_N$ be the set of all $N \times N$ (complex) Hermitian matrices, and let $\operatorname{tr} A = \sum_{i=1}^N a_{ii}$ denote the trace of a square matrix $A = (a_{ij})_{i,j=1}^N$. $\mathcal{H}_N$ is a real Hilbert space of dimension N^2 with respect to the symmetric bilinear form $(A, B) \mapsto \operatorname{tr} AB$. Let l_N denote the unique Lebesgue measure on $\mathcal{H}_N$ which satisfies the relation $l_N(Q) = 1$ for every cube $Q \subset \mathcal{H}_N$ with edges of length 1. A Gaussian probability measure on $\mathcal{H}_N$ invariant with respect to all orthogonal linear transformations of $\mathcal{H}_N$ is uniquely defined up to a scaling transformation. Such measures form a one-parameter family $(\mu_N^s)_{s>0}$, where the measure μ_N^s is specified by its density

$$g_N^s(A) = \frac{1}{(\sqrt{s2\pi})^{N^2}} \exp\left(-\frac{1}{2s}\ \operatorname{tr} A^2\right) \tag{1.1}$$

with respect to l_N. Thus, for a random matrix X distributed according to μ_N^s we have

$$E_{\mu_N^s} \operatorname{tr} X^2 = sN^2. \tag{1.2}$$

The set $\mathcal{H}_N$ endowed with the measure μ_N^s is called the Gaussian Unitary Ensemble (GUE). Let X be a random $N \times N$ Hermitian matrix

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(that is a random variable taking values in $\mathcal{H}_N$). We consider the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of the random matrix X as a finite sequence of *exchangeable* random variables. By definition, this means that their joint distribution P_N^X does not change under any permutation of these variables. Let for each $n, 1 \leq n \leq N$, $P_{n,N}^X$ denote the joint distribution of some n of these N variables. Obviously, $P_{n,N}^X$ is a permutation invariant probability measure in $\mathbb{R}^n$. In particular, the measure $P_{1,N}^X$ describes the distribution of a single eigenvalue. By definition, the *n-point correlation measure* $\rho_{n,N}^X$ of a random matrix X is a non-normalized measure defined by the relation

$$\rho_{n,N}^X = \frac{N!}{(N-n)!} P_{n,N}^X. \tag{1.3}$$

For a measurable set $A \subset \mathbb{R}^n$ the quantity $\rho_{n,N}^X(A)$ can be interpreted as the average number of $n-$tuples of eigenvalues in the set A. If the measure $\rho_{n,N}^X$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, its Radon-Nikodym derivative $R_{n,N}^X$ is called the *n-point correlation function* of the random matrix X. In particular, the measure $\rho_{1,N}^X$ has total mass N. For a measurable set $E \subset \mathbb{R}^1$, the quantity $\rho_{1,N}^X(E)$ expresses the expected number of the eigenvalues belonging to E. The corresponding density with respect to the Lebesgue measure in $\mathbb{R}^1$, if it exists, is called the *eigenvalue density* or the *density of states* (caution: under the same names the normalized versions of the same measures are considered in the literature as well). Let X_N be a random matrix with the distribution μ_N^s. For $n = 1, \ldots, N$ we set $P_{n,N}^{\mathrm{GUE},s} = P_{n,N}^{X_N}$ and $\rho_{n,N}^{\mathrm{GUE},s} = \rho_{n,N}^{X_N}$. A classical result for the GUE says that we have

$$P_{1,N}^{\mathrm{GUE},1/N} \underset{N\to\infty}{\to} W, \tag{1.4}$$

where the measures converge in the weak sense, and W is the standard Wigner measure on $[-2, 2]$ defined by the density

$$w(x) = (2\pi)^{-1}\sqrt{(4 - x^2)_+},\ x \in \mathbb{R}. \tag{1.5}$$

In terms of the correlation measures the same relation reads

$$\frac{1}{N}\rho_{1,N}^{\mathrm{GUE},1/N} \underset{N\to\infty}{\to} W. \tag{1.6}$$

For the $n-$point correlation measures we have a similar relation

$$\frac{1}{N^n}\rho_{n,N}^{\mathrm{GUE},1/N} \underset{N\to\infty}{\to} \underbrace{W \times \cdots \times W}_{n \text{ times}}, \tag{1.7}$$

which means that the eigenvalues become independent in the limit. However, for $n \geq 2$, the study of a finer asymptotics near a point from

the principal diagonal in the cube $(-2,2)^n$ shows [10, 11]: for every $u \in (-2,2)$ and $t_1, \dots, t_n \in \mathbb{R}^1$

(1.8)

$$\lim_{N\to\infty} \frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},\frac{1}{N}} \big(u + (t_1/Nw(u)), \dots, u + (t_n/Nw(u))\big) \\ = \det\left(\frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)}\right)_{i.j=1}^{n}.$$

This limit relation presents a pattern for many other results, in particular, for that of the present paper. The right hand side of this relation represents an example of the correlation function of a so-called determinantal (or fermionic) random point process [15]. In general the $n-$point correlation function R_n of such a process is given by the formula

$$R(u_1, \dots, u_n) = \det\big|K(u_i, u_j)\big|_{i,j=1}^{n},$$

where K is the kernel of an integral operator on the line, which is trace-class having been restricted to any finite interval in $\mathbb{R}$, and subject to some further conditions (see [15] for a detailed exposition). Moreover, in the asymptotic Hermitian random matrix theory $(K_N)_{N\geq 0}$ are the reproducing kernels of the subspaces of polynomials of degree $\leq N-1$ with respect to some weight on the line. In this case we call the corresponding matrix ensemble *determinantal.* The GUE gives an example of such an ensemble. To a large extent the asymptotic study of determinantal ensembles reduces to that of the respective kernels [3, 5]. In particular, the same local limit as in the case of GUE is established in [12, 3] for two broad classes of determinantal matrix ensembles. It is expected that the sin-kernel limit (first discovered by F. Dyson) is rather common. This is known as the *universality conjecture*. Outside the class of determinantal Hermitian random matrices only very few results on the asymptotics of the correlation functions are known (see, for instance, [10], where a mixture of determinantal measures is considered).

In the present paper we investigate the following non-determinantal ensemble of Hermitian random matrices. Let

(1.9) $$S_N^r = \{A \in \mathcal{H}_N : \operatorname{tr} A^2 = r^2\}$$

be the sphere in $\mathcal{H}_N$ of the radius $r > 0$ centered at the origin. Set $r = \sqrt{s}N$. The sphere $S_N^{\sqrt{s}N}$ carries a unique probability measure ν_N^s invariant with respect to all orthogonal linear transformations in the space $\mathcal{H}_N$. We call this measure the *fixed Hilbert-Schmidt norm ensemble* (or just HSE) to reserve the term "the fixed trace ensemble" for more

general ones (see [1]). Let Y_N be a random matrix distributed according to ν_N^s. We set for $n = 1, \dots, N$ $P_{n,N}^{\mathrm{HSE},s} = P_{n,N}^{Y_N}$ and $\rho_{n,N}^{\mathrm{HSE},s} = \rho_{n,N}^{Y_N}$. It is a well known result (see [11, 14]) that

$$P_{1,N}^{\mathrm{HSE},1/N} \underset{N\to\infty}{\to} W, \tag{1.10}$$

like in the case of GUE. In this paper we prove that the correlation functions $R_{n,N}^{\mathrm{HSE},1/N}$ of arbitrary order n ($1 \le n \le N$) near every point $u \in (-2, 2)$, have the same determinantal limit with the kernel $\sin \pi(t_1 - t_2)/\pi(t_1 - t_2)$ as the GUE correlation functions (for $n = 1$ the limit equals 1). More precisely, we establish in this paper the following result.

Theorem. *Let $R_{n,N}^{\nu,1/N}$ be the $n-$point correlation function of the eigenvalues for a random matrix uniformly distributed on the sphere $S_N^{\sqrt{N}}$. Then for every $u \in (-2, 2)$ and $t_1, t_2, \dots, t_n \in \mathbb{R}^1$*

$$\frac{1}{(Nw(u))^n} R_{n,N}^{\nu,1/N}\left(u + \tfrac{t_1}{Nw(u)}, \dots, u + \tfrac{t_n}{Nw(u)}\right) \to \det\left(\frac{\sin \pi(t_i - t_j)}{\pi(t_i - t_j)}\right)_{i,j=1}^n \tag{1.11}$$

as $N \to \infty$. For every $\alpha \in (0, 1)$ and $A > 0$ the relation (1) *holds uniformly in all $u \in [-2+\alpha, 2-\alpha]$, and $t_1 \in [-A, A], \dots, t_n \in [-A, A]$.*

The class of fixed trace matrix ensembles was studied in several recent publications. The authors are indebted to G. Akemann for drawing their attention, after the first version of the present work has appeared as a preprint, to the paper [2], he published jointly with G. Vernizzi. In this paper the local universality is studied for a class of ensembles containing the HSE. The authors provide heuristic arguments for the universality near zero based on the complex inversion of the Laplace transform. Our proof is based on the complex Laplace inversion as well (more precisely, we arrive at the Fourier transform after a sequence of translations of the origin and rescalings depending on N). However, establishing bounds for orthogonal functions and related kernels in the complex domain allows us to rigorously conclude convergence near every point of the interval $(-2, 2)$.

In the following we will sketch the main steps of the proof. The guiding principle is that results for the Hilbert-Schmidt ensembles are deducible from the corresponding results for the GUE using the 'equivalence of ensembles' or the 'concentration phenomenon'. In our setup a primitive form of the concentration is given by the law of large numbers for the squares of the Hilbert-Schmidt norms of the GUE random matrices. Supplemented by some estimates of the probabilities of large

deviations, this is the main tool in [9], where the universality is shown near zero for the correlation measures (rather than the correlation functions) of the Hilbert-Schmidt ensemble; convergence was shown there in the weak topology determined by the continuous compactly supported functions. Such simple arguments seem to be insufficient for proving local results for correlation functions of eigenvalues (maybe, with exception for eigenvalues near zero), particularly, for stronger topologies. This agrees with M.L.Mehta's doubts ([11], Sect. 27.1, p.490) concerning the deducibility of the local results for correlation functions of the Hilbert-Schmidt ensembles from the corresponding results for GUE by using the equivalence of ensembles. Solving this open problem in the present paper, we use a *local form* of concentration given by the local central limit theorem for the densities of the squared Hilbert-Schmidt norms. First we represent the Hilbert-Schmidt measure as a conditional measure of the GUE, given the Hilbert-Schmidt norm of the GUE random matrix. Starting with the disintegration of the GUE according to the level sets of the Hilbert-Schmidt norm, we arrive at formula (2.9) which is the crucial ingredient of the proof. Here we have to extend the scaling parameter to the complex domain. The formal Fourier inversion applied to this formula gives an 'heuristic proof' of the result. For an outline of it see Section 2. To make this sketch rigorous we need asymptotic estimates in the complex domain for the kernels related to the Hermite functions. This is done in Section 3, based on the results in [3], [4], [5]. Unfortunately, some of the results we need are contained in these papers not explicitly enough and have to be extracted from the proofs rather than from the statements (see the proof of Lemma 3). The main convergence result we need is Lemma 3. All necessary bounds are summarized in Proposition 1. Based on these two facts, we complete the proof. Note that the analytic part of the present paper may be viewed as a form of the Tauberian theorem.

The authors are indebted to the anonymous referees for careful reading of the manuscript. Moreover, the main theorem in the first version of the paper required excluding of a neighborhood of zero (this case has been considered in the note [9]). Stimulating questions of one of the referees and the editor led us to a revision of the proof which removes this restriction. Finally, we would like to thank Justine Swierkot for her assistance in the preparation of the text.

2. Disintegration, a Fourier transform formula and the sketch of the proof

In this section we discuss a disintegration representation of the GUE in terms of the HSE, and derive a Fourier transform formula involving these matrix ensembles. We suppress in this section the "spectral" arguments of the correlation functions and related quantities assuming that these arguments vary inside the domain described in Section 1.

For every $r > 0$ denote by S_N^r the sphere of radius r in $\mathcal{H}_N$ centered at the origin. Let for $s > 0$ X_N be a random matrix in $\mathcal{H}_N$ distributed according to $\mu_N^{\mathrm{GUE},s}$. Set $T_N = \operatorname{tr}(X_N^2/s)$ and $Y_N = NX_N/\sqrt{\operatorname{tr} X_N^2}$. The random variable T_N can be represented as a sum of N^2 squares of independent standard Gaussian random variables, hence it follows the familiar $\chi^2_{N^2}$ distribution. Moreover, T_N and Y_N are independent, and Y_N is uniformly distributed on the sphere S_N^N in $\mathcal{H}_N$, that is, Y_N is distributed according to ν_N^1 in the notation of the previous section. Then X_N can be represented as

$$X_N = \frac{Y_N}{N}\sqrt{sT_N} \tag{2.1}$$

with T_N and Y_N as above. Let γ_{N^2} denote the the probability density of T_N. Then it follows from (2.1) that

$$\mu_N^s = \int_0^\infty \nu_N^{u/N^2}\gamma_{N^2}(s^{-1}u)s^{-1}du. \tag{2.2}$$

As a consequence of (2.2), the correlation functions of GUE and HSE for $1 \le n \le N-1$ satisfy the relation

$$R_{n,N}^{\mathrm{GUE},s} = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2}\gamma_{N^2}(s^{-1}u)s^{-1}du. \tag{2.3}$$

Note that

$$ET_N = N^2,\ DT_N^2 = E(T_N - ET_N)^2 = 2N^2,$$

and, for every $m > 0$,

$$\gamma_m(u) = \begin{cases} (2^{m/2}\Gamma(m/2))^{-1}x^{(m/2)-1}e^{-x/2}, \text{ if } u \ge 0, \\ 0, \text{if } u < 0. \end{cases} \tag{2.4}$$

Set for $s > 0$ $\gamma_{m,s}(\cdot) = s^{-1}\gamma_m(s^{-1}\cdot)$, so that $\gamma_{m,1} = \gamma_m$ (note that the same set of densities with a different parametrization appears in Lemma 8 as $(f_{a,p})_{a,p>0}$). Observe now that the probability density of sT_N is given by $\gamma_{N^2,s}(\cdot)$. Then (2.3) can be rewritten as

$$R_{n,N}^{\mathrm{GUE},s} = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2}\gamma_{N^2,s}(u)du. \tag{2.5}$$

In particular, we have

$$R_{n,N}^{\mathrm{GUE},1/N} = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/N}(u)du. \tag{2.6}$$

Our goal is to investigate the limiting behavior of $R_{n,N}^{\mathrm{HSE},1/N}$ when n is fixed, $N \to \infty$, and the "spectral" arguments of $R_{n,N}^{\mathrm{HSE},1/N}$ vary within an $\epsilon/N-$neighborhood of a point from the main diagonal in the cube $(-2,2)^n$. However, we prefer to consider the function $u \mapsto R_{n,N}^{\mathrm{HSE},u/N^2}$ rather than its value $R_{n,N}^{\mathrm{HSE},1/N}$ at N. Moreover, we will perform the study of the latter function indirectly, first dealing with the product $u \mapsto R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/N}(u)$. After the change of variable $(u-N)/\sqrt{2} = v$ in (2.6) we obtain the relation

$$R_{n,N}^{\mathrm{GUE},1/N} = \int_{-\infty}^\infty q_{N^2}(v)dv, \tag{2.7}$$

where

$$q_{N^2}(v) = R_{n,N}^{\mathrm{HSE},1/N+v\sqrt{2}/N^2} \sqrt{2}\gamma_{N^2,1/N}(N+v\sqrt{2}). \tag{2.8}$$

Notice that $v \mapsto \sqrt{2}\ \gamma_{N^2,1/N}(N+v\sqrt{2})$ is the probability density of the centered and normalized random variable $(T_{N^2} - N^2)/\sqrt{2N^2}$, and it tends to the standard normal density $\varphi : v \mapsto (1/\sqrt{2\pi}) \exp(-v^2/2)$ as $N \to \infty$. Thus, the limit behavior of the density $\gamma_{N^2,1/N}(N + \cdot\sqrt{2})$ is well understood, and we have to study $q_{N^2}(\cdot)$. For every fixed u, due to the relation $\gamma_{N^2,s}(\cdot) = s^{-1}\gamma_{N^2}(s^{-1}\cdot)$, we can analytically extend $\gamma_{N^2,s}(u)$ to the the domain $\Re s > 0$. Moreover, in the formula (2.5) the integral in the right hand side can be analytically continued in s in accordance with the continuation of $\gamma_{N^2,s}(u)$ mentioned above. This leads to the corresponding continuation of $R_{n,N}^{\mathrm{GUE},s}$ so that (2.5) holds for s from the right half-plane. Now we will evaluate the Fourier transform of the (nonprobabilistic) density q_{N^2} keeping in mind that by (2.7) and (2.8) it is a nonnegative integrable function.

Lemma 1.

$$\int_{-\infty}^\infty \exp(ipv) q_{N^2}(v)dv = \phi_{N^2}(p) R_{n,N}^{GUE,1/((1-ip\sqrt{2}/N)N)}, \tag{2.9}$$

where

$$\phi_{N^2}(p) = \exp(-ipN/\sqrt{2})(1-ip\sqrt{2}/N)^{-(N^2/2)} \tag{2.10}$$

is the characteristic function of the random variable $(T_{N^2} - N^2)/\sqrt{2N^2}$.

Proof. Denoting by C_m the normalizing constant in formula (2.4), we have
(2.11)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp(ipv) q_{N^2}(v) dv \\
= &\int_{-\infty}^{\infty} \exp(ipv) R_{n,N}^{\mathrm{HSE},1/N+v\sqrt{2}/N^2} \sqrt{2}\gamma_{N^2,1/N}(N+v\sqrt{2}) dv \\
= &\int_{-\infty}^{\infty} \exp\left(\frac{ip(u-N)}{\sqrt{2}}\right) R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/N}(u) du \\
= &\exp\left(\tfrac{-ipN}{\sqrt{2}}\right) \int_{-\infty}^{\infty} \exp(ipu\sqrt{2}) R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/N}(u) du \\
= &\exp\left(\tfrac{-ipN}{\sqrt{2}}\right) \int_{-\infty}^{\infty} R_{n,N}^{\mathrm{HSE},u/N^2} C_{N^2} N (Nu)^{\frac{N^2}{2}-1} \exp\left(-\tfrac{Nu}{2}\Big(1-\tfrac{ip\sqrt{2}}{N}\Big)\right) du \\
= &\exp\left(\tfrac{-ipN}{\sqrt{2}}\right) \left(1-\tfrac{ip\sqrt{2}}{N}\right)^{-\frac{N^2}{2}+1} \times \\
&\int_{-\infty}^{\infty} R_{n,N}^{\mathrm{HSE},u/N^2} C_{N^2} N \left(Nu\left(1-\tfrac{ip\sqrt{2}}{N}\right)\right)^{\frac{N^2}{2}-1} \exp\left(-\tfrac{Nu}{2}\left(1-\tfrac{ip\sqrt{2}}{N}\right)\right) du \\
= &\exp\left(\tfrac{-ipN}{\sqrt{2}}\right) \left(1-\tfrac{ip\sqrt{2}}{N}\right)^{-\frac{N^2}{2}} \int_{-\infty}^{\infty} R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,1/((1-ip\sqrt{2}/N)N)}(u) du \\
= &\phi_{N^2}(p) R_{n,N}^{\mathrm{GUE},1/((1-ip\sqrt{2}/N)N)}.
\end{aligned}$$

□

In the following we will outline our approach. Write

$$\frac{1}{(Nw(u))^n} q_{N^2}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{N^2}(p) \frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/((1-ip\sqrt{2}/N)N)} dp.$$

Passing to the limit in the integral on the right hand side (uniformly with respect to the spectral variables), the right hand side has the same limit as

$$\frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/N} \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_{N^2}(p) dp$$

or, in view of the local Central Limit Theorem (CLT) , as

$$\frac{1}{\sqrt{2\pi}(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/N}.$$

On the other hand, it follows from(2.8) that

$$\frac{1}{(Nw(u))^n} q_{N^2}(0) = \frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{HSE},1/N} \sqrt{2}\gamma_{N^2,1/N}(N). \tag{2.12}$$

Again, the local CLT implies that $\sqrt{2}\gamma_{N^2,1/N}(N) \to 1/\sqrt{2\pi}$. Therefore,

$$\frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{HSE},1/N}$$

tends to the same limit as

$$\frac{1}{(Nw(u))^n} R_{n,N}^{\mathrm{GUE},1/N},$$

and the conclusion follows. In the next Section these heuristic arguments will be made rigorous.

3. Proofs

For every $\alpha \in \mathbb{R}$, through the rest of the paper, we will denote by $(\cdot)^\alpha$ the function

$$(\cdot)^\alpha : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C} : z \mapsto \exp \alpha \log z, \tag{3.1}$$

where log denotes the principal branch of the logarithm. We use $\sqrt{\cdot}$ as a notation for $(\cdot)^{1/2}$ extended to 0 by $\sqrt{0} = 0$.

The same symbol (for example, $C, L, M, R, \alpha, \dots, \delta$ with or without indices, N_0, and so on) may denote different constants in the bounds obtained in the rest of the paper (even in the same proof). However, we will indicate explicitly the dependence of such constants on parameters.

First we are going to state some definitions and basic formulas related to the Hermite polynomials and Hermite functions. Then we will formulate some results on the asymptotics of the Hermite polynomials and Hermite functions in the complex plane. The asymptotic behavior of the Hermite polynomials at the scale we are interested in was first established in 1922 by Plancherel and Rotach. For a convenient form of these (and much more general) results we refer to the monograph [5] and the papers [3],[4] (in particular, Appendix B), and [6]. These results will be employed later in this section.

Let $s > 0$, and let for every $N, N \geq 0$, $\tilde{p}_N(\cdot, s)$ be a polynomial of degree N with a positive leading coefficient such that for $(\tilde{p}_N(\cdot, s))_{N\geq 0}$ the relations

$$\int_{\mathbb{R}} \tilde{p}_M(x, s)\tilde{p}_N(x, s) \exp\left(-x^2/(2s)\right) dx = \delta_{M,N}, \;\; M, N = 0, 1, \dots,$$

hold.

The (monic) Hermite polynomials $(\tilde{H}_N(\cdot, s))_{N=0,1,\dots}$ are defined by the expansion

$$\exp\left(\gamma x - \gamma^2 s/2\right) = \sum_{N=0}^{\infty} \tilde{H}_N(x, s)\frac{\gamma^N}{N!} \tag{3.2}$$

and for $M, N = 0, 1, \dots$ satisfy the relations

$$\int_{\mathbb{R}} \tilde{H}_M(x,s)\tilde{H}_N(x,s) \exp\left(-x^2/(2s)\right) dx = \delta_{M,N}\sqrt{2\pi} s^{(M+N+1)/2} N!$$

so that

$$\tilde{p}_N(\cdot, s) = \frac{1}{(2\pi)^{1/4} s^{(2N+1)/4} (N!)^{1/2}} \tilde{H}_N(\cdot, s),\ \ N = 0, 1, \dots$$

The polynomials $\tilde{p}_N(\cdot, s)$ satisfy the difference equations

(3.3)

$$x\, \tilde{p}_N(x,s) = s^{1/2}\sqrt{N+1}\, \tilde{p}_{N+1}(x,s) + s^{1/2}\sqrt{N}\, \tilde{p}_{N-1}(x,s),\ \ N = 1, 2, \dots$$

The Hermite functions $(\tilde{\varphi}_N(\cdot, s))_{N=0,1,\dots}$ defined by

$$\tilde{\varphi}_N(x,s) \ = \tilde{p}_N(x,s) \exp\left(-x^2/(4s)\right),\ N = 0, 1, \dots,$$

form an orthonormal sequence of functions in $L_2(\mathbb{R}, \lambda)$ where λ is the Lebesgue measure in $\mathbb{R}$. Let us introduce the standardized Hermite polynomials and Hermite functions (corresponding to the weight $\exp\left(-x^2/2\right)$) by the relations

$$H_N(\cdot) = \tilde{H}_N(\cdot, 1),\ p_N(\cdot) = \tilde{p}_N(\cdot, 1),\ \varphi_N(\cdot) = \tilde{\varphi}_N(\cdot, 1)$$

so that

$$\tilde{H}_N(\cdot, s) = s^{N/2} H_N(\cdot\, s^{-1/2}),\ \tilde{p}_N(\cdot, s) = s^{-1/4} p_N(\cdot\, s^{-1/2}), \\ \tilde{\varphi}_N(\cdot, s) = s^{-1/4} \varphi_N(\cdot\, s^{-1/2}). \tag{3.4}$$

The above relations for the Hermite polynomials imply that the Hermite functions satisfy, for every $k \geq 1$, the following system of differential equations:

$$\begin{aligned} \varphi_k'(x) &= -\frac{x}{2}\varphi_k(x) + \sqrt{k}\varphi_{k-1}(x), \\ \varphi_{k-1}'(x) &= -\sqrt{k}\varphi_k(x) + \frac{x}{2}\varphi_{k-1}(x). \end{aligned} \tag{3.5}$$

The reproducing kernel $\tilde{K}_N(\cdot, \cdot, s)$ of the orthogonal projection in $L_2(\mathbb{R}, \lambda)$ onto the linear span of $\tilde{\varphi}_0(\cdot, s), \dots, \tilde{\varphi}_{N-1}(\cdot, s)$ is given by

$$\tilde{K}_N(x, y, s) = \sum_{k=0}^{N-1} \tilde{\varphi}_k(x,s)\tilde{\varphi}_k(y,s). \tag{3.6}$$

Note that the correlation functions of the GUE can be expressed in terms of the latter kernel as

$$R_{n,N}^{\mathrm{GUE},s}(x_1, \dots, x_n) = \det\Big(\tilde{K}_N(x_i, x_j, s)\Big)_{i,j=1}^{n}. \tag{3.7}$$

(see [11], (6.2.7)).

Further, setting

$$K_N(x,y) = \tilde{K}_N(x,y,1),$$

we obtain

$$\tilde{K}_N(x,y,s) = s^{-1/2} K_N(xs^{-1/2}, ys^{-1/2}). \tag{3.8}$$

The Christoffel - Darboux formula for the kernels $\tilde{K}_N$ and K_N reads

$$\tilde{K}_N(x,y,s) = \frac{\tilde{\varphi}_N(x,s)\tilde{\varphi}_{N-1}(y,s) - \tilde{\varphi}_{N-1}(x,s)\tilde{\varphi}_N(y,s)}{x-y} \tag{3.9}$$

and

$$K_N(x,y) = \frac{\varphi_N(x)\varphi_{N-1}(y) - \varphi_{N-1}(x)\varphi_N(y)}{x-y}. \tag{3.10}$$

Note that the kernels $\tilde{K}_N, K_N$ enjoy the property

$$\tilde{K}_N(-z_1,-z_2,s) = \tilde{K}_N(z_1,z_2,s),\ K_N(-z_1,-z_2) = K_N(z_1,z_2). \tag{3.11}$$

As to K_N, this is a consequence of (3.10) and the fact that the functions $(\varphi_N(\cdot))_{N\geq 0}$ are even or odd depending on N being even or odd; for $\tilde{K}_N$ it follows from (3.8). The following integral representation of the reproducing kernel is a version of formula (4.56) in [8]:

$$\begin{aligned} &K_N(x,y) = \\ &\sqrt{\frac{N}{2}} \int_0^\infty (\varphi_N(x+\tau)\varphi_{N-1}(y+\tau) + \varphi_{N-1}(x+\tau)\varphi_N(y+\tau))d\tau \end{aligned} \tag{3.12}$$

so that

(3.13)

$$\begin{aligned} &\tilde{K}_N(x,y,s) = \\ &\sqrt{\frac{N}{2s}} \int_0^\infty \Big(\varphi_N\big(\tfrac{x}{\sqrt{s}}+\tau\big)\varphi_{N-1}\big(\tfrac{y}{\sqrt{s}}+\tau\big) + \varphi_{N-1}\big(\tfrac{x}{\sqrt{s}}+\tau\big)\varphi_N\big(\tfrac{y}{\sqrt{s}}+\tau\big)\Big)\, d\tau. \end{aligned}$$

The relations (3.4), (3.8), and our extension of the function $(\cdot)^\alpha$ to $\mathbb{C}\setminus(-\infty,0)$ allow us to continue $\tilde{H}_N(x,\cdot)$, $\tilde{p}_N(x,\cdot)$, $\tilde{\varphi}_N(x,\cdot)$ and $\tilde{K}_N(x,y,\cdot)$ to this domain analytically in the parameter s. The relations (3.2)-(3.11) remain valid under these continuations. So does (3.13) whenever the integrals there are well-defined.

Denote by w the analytic continuation of the standard Wigner density (1.5) to the domain $\mathbb{C}\setminus\big((-\infty,2]\cup[2,\infty)\big)$. Let us define for every $\alpha\in(0,2)$ and $\beta>0$ the sets $S_{\alpha,\beta}$ and $\overline{S_{\alpha,\beta}}$ by the relations

$$S_{\alpha,\beta} = \{z\in\mathbb{C} : |\Re z| < 2-\alpha, |\Im u| < \beta\}$$

and

$$\overline{S_{\alpha,\beta}} = \{z\in\mathbb{C} : |\Re z| \leq 2-\alpha, |\Im u| \leq \beta\}.$$

Set for $H \in \mathbb{R}$ $d(H) = \sqrt{1+iH}$ and observe that

$$|d(H)| = (1+H^2)^{1/4}, \tag{3.14}$$

$$\Re d(H) = \sqrt{\frac{\sqrt{1+H^2}+1}{2}}, \tag{3.15}$$

$$\Im d(H) = \operatorname{sgn} H \sqrt{\frac{\sqrt{1+H^2}-1}{2}}, \tag{3.16}$$

and

$$\big(\Re d(H)\big)^2 - \big(\Im d(H)\big)^2 = 1. \tag{3.17}$$

It is clear from (3.17) that

$$|\Re(ud(H))| \geq |\Im(ud(H))| \tag{3.18}$$

for every $u \in \mathbb{R}$. Note that for every $b \geq 0$ the equation

$$|\Im d(H)| = b$$

has a unique nonnegative solution

$$H_b = \sqrt{(1+2b^2)^2 - 1}. \tag{3.19}$$

Lemma 2. *Let α, β, and u be real numbers such that $\alpha \in (0,1)$, $0 < \beta < \alpha$, and $u \in (-2+2\alpha, 2-2\alpha)$. Then the following assertions hold true:*

1. *For every $b \in [-\infty, \infty]$ the relations $|H| < |H_b|$ and $|\Im\big(d(H)\big)| < b$ are equivalent (we set here $H_\infty = \infty$, $\Im\big(d(\infty)\big) = \infty$, and $\Im\big(d(-\infty)\big) = -\infty$); the relations $|H| \leq |H_b|$ and $|\Im\big(d(H)\big)| \leq b$ are also equivalent.*
2. *For every real H we have $ud(H) \in \overline{S_{\alpha,\beta}}$ whenever $|\Im(ud(H))| \leq \beta$;*
3. *If for a certain real H the relation $|H| \leq |H_{\beta/2}|$ (or, equivalently, $|\Im\big(d(H)\big)| \leq \beta/2$) holds, it follows that $ud(H) \in \overline{S_{\alpha,\beta}}$.*

Proof. It is clear from (3.19) and the definition of H_b that for $b \in [0,\infty]$ and $H \in [0,\infty]$ the functions $b \mapsto H_b$ and $H \mapsto |\Im d(H)|$ are strictly increasing mutually inverse mappings. This shows equivalence of the conditions in point 1 of the lemma for the functions in question restricted to $[0,\infty]$. The same is true for $b, H \in [-\infty,\infty]$ since both functions are even. This proves assertion 1.
For every H such that

$$\big|\Im\big(ud(H)\big)\big| \leq \beta \tag{3.20}$$

we have

$$(3.21) \qquad \left|\Re\big(ud(H)\big)\right| < 2-\alpha$$

since it follows from the relation (3.17) and the assumptions $\beta \le \sqrt{\alpha}$, $u \in (-2+2\alpha, 2-2\alpha)$ that

$$\begin{aligned}\left|\Re\big(ud(H)\big)\right| = \sqrt{u^2 + \big(\Im u\big(d(H)\big)\big)^2} &\le \sqrt{(2-2\alpha)^2+\beta^2}\\ &< \sqrt{(2-2\alpha)^2+\alpha^2} < 2-\alpha.\end{aligned}$$

In view of the definition of $\overline{S_{\alpha,\beta}}$, the second assertion is proved. The third assertion is an immediate consequence of the second one since $|u| \le 2$.

□

Corollary 1. *Let α, β, and u satisfy the assumptions of Lemma 2, and assume that the relation $ud(H) \notin \overline{S_{\alpha,\beta}}$ holds for some real H. Then we have $|\Im(ud(H))| > \beta$. Moreover, the relation $|\Im(ud(H'))| > \beta$ holds for every real H' such that $|H'| \ge |H|$.*

Proof. The corollary follows immediately from the second and the first assertions of Lemma 2 combined with the fact that, by (3.16), $|\Im d(H)|$ is a strictly increasing function of $|H|$.

□

In the next Lemma the main result on convergence is formulated.

Lemma 3. *For every number $\alpha \in (0,1)$ there exists a positive number $\overline{H} = \overline{H}(\alpha)$ such that for every $A > 0$ the relation*

$$(3.22) \qquad \begin{aligned}&\frac{1}{Nw(u)}\tilde{K}_N\left(\left(u+\frac{t_1}{Nw(u)}\right)d(H), \left(u+\frac{t_2}{Nw(u)}\right)d(H), N^{-1}\right)\\ &\underset{N\to\infty}{\longrightarrow} \frac{\sin\big(\pi(t_1-t_2)d(H)w(ud(H))/w(u)\big)}{\pi(t_1-t_2)d(H)}\end{aligned}$$

holds uniformly for all real numbers u, t_1, t_2, and H provided that $u \in [-2+2\alpha, 2-2\alpha]$, $|t_1| \le A$, $|t_2| \le A$, and $|H| \le \overline{H}$.

Proof. Note that for every $\alpha \in (0,1)$ there exists $\beta' = \beta'(\alpha) > 0$ such that for every $A > 0$ the relation

$$(3.23) \qquad \frac{1}{Nw(v)}\tilde{K}_N\left(v+\frac{z_1}{Nw(v)}, v+\frac{z_2}{Nw(v)}, \frac{1}{N}\right) \underset{N\to\infty}{\longrightarrow} \frac{\sin\pi(z_1-z_2)}{\pi(z_1-z_2)}$$

holds uniformly for all complex numbers $v \in \overline{S_{\alpha,\beta'(\alpha)}}$, z_1, and z_2 such that $|z_1| \le A, |z_2| \le A$.
This assertion (and more general ones involving some class of weights)

is implicitly contained in the papers [3] and [4] (see also the monograph [5]). Actually, Lemma 6.1 in [3] establishes the desired result for real u, z_1 and z_2. However, the same reasoning applies to the complex u, z_1, and z_2 satisfying the assumptions just made provided that β' is chosen to be sufficiently small in accordance with α. The key ingredient of the proof in these references, the boundedness property of a certain derivative, is established in [3] and [4] for some complex neighborhood of an arbitrary real point $u \in [-2+2\alpha, 2-2\alpha]$ (see relation (4.122) in [3]) which allows to bound this derivative in a rectangular strip $\overline{S_{\alpha,\beta'(\alpha)}}$. Lowering β' if necessary, we can substitute $\beta'(\alpha)$ by some $\beta(\alpha) \le \alpha/4$. Note that the function $w(\cdot)$ has no zeroes in $\mathbb{C} \setminus \big((-\infty, 2] \cup [2, \infty)\big) \supset \overline{S_{\alpha,\beta(\alpha)}}$. Picking a point $v' \in \overline{S_{\alpha,\beta(\alpha)}}$ and setting in (3.23) $z_i = z_i' \, w(v)/w(v')$ $(i = 1, 2)$, we obtain
(3.24)

$$\frac{1}{Nw(v')}\tilde{K}_N\left(v + \frac{z_1'}{Nw(v')}, v + \frac{z_2'}{Nw(v')}, \frac{1}{N}\right) \underset{N\to\infty}{\longrightarrow} \frac{\sin\left(\frac{\pi(z_1'-z_2')w(v)}{w(v')}\right)}{\pi(z_1'-z_2')}.$$

Note that for $v \in \overline{S}_{\alpha,\beta(\alpha)}$ we have $C^{-1} \le |w(v)| \le C$ with some $C > 1$. This fact and (3.23) imply that for every $A > 0$ (3.24) holds uniformly in $v, v' \in \overline{S_{\alpha,\beta(\alpha)}}$ and $|z_1'| \le A, |z_2'| \le A$.

According to assertion 3 of Lemma 2, for every $u \in [-2+2\alpha, 2-2\alpha]$ and every $H \in [-H_{\beta(\alpha)/2}, H_{\beta(\alpha)/2}]$ we have $ud(H) \in \overline{S_{\alpha,\beta(\alpha)}}$. More, it is clear from (3.14) that for $H \in [-H_{\beta(\alpha)/2}, H_{\beta(\alpha)/2}]$ we have ${C'}^{-1} < |d(H)| < C'$ with some positive constant $C' > 1$ depending on α. Then, setting $v' = u \in [-2+2\alpha, 2-2\alpha]$, $v = ud(H) \in \overline{S_{\alpha,\beta(\alpha)}}$, $z_i' = t_i d(H)$ $(i = 1, 2)$, we obtain (3.22).

□

Remark 1. It is sometimes convenient to use the following reformulation of (3.23) and the related assumptions :
For every number $\alpha \in (0, 1)$ there exists a real numbers $\beta = \beta(\alpha) \le \alpha/4$ such that for every $A > 0$ the relation

$$\left|\frac{1}{N}\tilde{K}_N\big(v_1, v_2, N^{-1}\big) - \frac{\sin\big(N\pi(v_1-v_2)w(v)\big)}{N\pi(v_1-v_2)}\right| \underset{N\to\infty}{\longrightarrow} 0$$

holds uniformly for all complex numbers v, v_1, v_2 satisfying the constraints $v \in \overline{S_{\alpha,\beta(\alpha)}}$, $|v_1 - v| \le A/N$, $|v_2 - v| \le A/N$.

We will now derive upper bounds for the modulus of the kernel

$$\frac{1}{N}\tilde{K}_N\big(\cdot, \cdot, N^{-1}\big)$$

valid for various values of its arguments. We will use asymptotics of two types for the Hermite functions: one of them is valid in a narrow strip around the interval $[-2+\alpha, 2-\alpha]$; another one applies in the complement to some neighborhood of $[-2,+2]$. With this asymptotics, after obtaining certain intermediate bounds for the quantity $|N^{-1}\tilde{K}_N\big(u+t_1/N)d(H), u+t_2/N)d(H), N^{-1}\big)|$, we will finally determine for this quantity an upper estimate which does not depend on $H \in \mathbb{R}$. This bound and analogous bounds for the correlation functions will be derived in Proposition 1.
In the next two lemmas we present upper estimates for the Hermite functions. These estimates are immediate consequences of the Plancherel-Rotakh-type asymptotics. First we consider a bound valid in the strip $\overline{S}_{\alpha,\beta(\alpha)}$ where $\beta(\alpha)$ was defined in the proof of Lemma 3.

Lemma 4. *For any $\alpha \in (0,1)$ there exist a constant $C(\alpha) > 0$ such that we have*

$$\big|\varphi_N(z\sqrt{N})\big| \leq \big(C(\alpha)\big)^N \tag{3.25}$$

and

$$\big|\varphi_{N-1}\big(z\sqrt{N}\big)\big| \leq \big(C(\alpha)\big)^{N-1} \tag{3.26}$$

for every $N \in \mathbb{N}$ and $z \in \overline{S}_{\alpha,\beta(\alpha)}$.

Proof. First we recall a known result about the Plancherel-Rotach asymptotics for the Hermite functions in a rectangular strip (Theorem 2.2, part (ii), in [4]; the particular case of the Hermite functions for the measure $\exp(-x^2)dx$ is considered there in Appendix B). This strip in the complex plane contains, up to some neighborhoods of the ends, the interval where the zeroes of the Hermite functions asymptotically concentrate. Set

$$\psi : \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty)) \to \mathbb{C} : z \mapsto \frac{1}{2\pi}(1-z)^{1/2}(1+z)^{1/2}.$$

The function arcsin is defined as the inverse function of

$$\sin : \left\{ z \in \mathbb{C} : |\Re(z)| < \frac{\pi}{2} \right\} \to \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty)).$$

In our notation, the statement in [4] reads: *for every $\alpha \in (0,1)$ uniformly in $z \in \{z' \in \mathbb{C} : |\Re(z')| \leq 1-\alpha, |\Im(z')| \leq \alpha\}$ we have*

$$\begin{aligned} &\tilde{\varphi}_N(\sqrt{2N}z, 1/2) \\ &\quad = \sqrt{\frac{2}{\pi\sqrt{2N}}}(1-z)^{-1/4}(1+z)^{-1/4} \\ &\qquad \times \left\{ \cos\left(N\pi \int_1^z \psi(y)dy + \frac{1}{2}\arcsin z\right)\left(1 + O\left(\frac{1}{N}\right)\right) \right. \\ &\qquad\qquad \left. + \sin\left(N\pi \int_1^z \psi(y)dy - \frac{1}{2}\arcsin z\right) O\left(\frac{1}{N}\right)\right\}. \end{aligned} \tag{3.27}$$

The analytic functions $z \mapsto \int_1^z \psi(y)dy$ and arcsin are bounded on the set $\{z' \in \mathbb{C} : |\Re(z')| \leq 1-\alpha, |\Im(z')| \leq \alpha\}$. Representing cos and sin as the sum and the difference of the exponentials, it follows from (3.27) (omitting the multiplier $N^{-1/4}$) that the modulus of the left hand side of (3.27) is bounded above by the $N-$th power of a positive number depending on α. For every $\alpha \in (0,2)$, due to the relations

$$\varphi_N(z\sqrt{N}) = 2^{-1/4}\tilde{\varphi}_N\left(\frac{z}{\sqrt{2}}\sqrt{N}, 1/2\right) = 2^{-1/4}\tilde{\varphi}_N\left(\frac{z}{2}\sqrt{2N}, 1/2\right),$$

a similar bound by the $N-$th power of some number depending on α holds for $\varphi_N(z\sqrt{N})$ uniformly for $z \in \overline{S}_{\alpha,\alpha}$. For

$$\varphi_{N-1}(z\sqrt{N}) = \varphi_{N-1}\big(z\sqrt{(N-1)}\sqrt{N/(N-1)}\big)$$

we also have a bound of such type in $\overline{S}_{\alpha,\alpha/4} \subset \overline{S}_{\alpha/2,\alpha/2}$. Indeed, set $\rho_N = \sqrt{N/(N-1)}$, and let $N_0 = N_0(\alpha)$ be defined by the equation $\rho_{N_0} = (2-\alpha/2)/(2-\alpha)$. Then we have $\rho_N \leq 2$ and $\rho_N\overline{S}_{\alpha,\alpha/4} \subset \overline{S}_{\alpha/2,\alpha/2}$ whenever $N \in \mathbb{N}$ satisfies $N \geq N_0$. This means that for $z \in \overline{S}_{\alpha,\alpha/4}$ both $z \in \overline{S}_{\alpha/2,\alpha/2}$ and $z\sqrt{N/(N-1)} \in \overline{S}_{\alpha/2,\alpha/2}$ hold for every $N \geq N_0$. Therefore, the conclusion of the Lemma is valid for such values of N. Choosing a larger constant $C(\alpha)$ if necessary and recalling that $\beta(\alpha) \leq \alpha/4$, the proof is completed.

□

We will obtain now some estimates valid the Hermite functions in the domain $|\Im z| \geq \delta > 0$.

Lemma 5. *For every $\delta > 0$ there exist constants $C(\delta)$ and $M(\delta)$ such that the inequalities*

$$\left|\varphi_N(z\sqrt{N})\right| \leq C(\delta)N^{-1/4}M^N(\delta)\,|z|^N \tag{3.28}$$

and

$$\left|\varphi_{N-1}(z\sqrt{N})\right| \le C(\delta)(N)^{-1/4}M^{N-1}(\delta)\,|z|^{N-1} \tag{3.29}$$

hold for every $N \in \mathbb{N}$ and every z satisfying $|\Re z| \ge |\Im z| \ge \delta$.

Proof. Again we start with stating some known result about the Plancherel-Rotach asymptotics for the Hermite polynomials in the complex plane (Theorem 7.185 in [5], see also [18] and the references therein). In our notation, uniformly in z from every compact set contained in $(\mathbb{C} \cup \{\infty\}) \setminus [-\sqrt{2}, \sqrt{2}]$, we have

(3.30)

$$\tilde{H}_N(z,(2N)^{-1}) = \frac{U(z)}{2}\frac{\exp\left(N(z-\sqrt{z^2-2})^2/4\right)}{(z-\sqrt{z^2-2})^N}\left(1+O\left(\frac{1}{N}\right)\right),$$

where

$$U(z) := \left(\frac{z-2}{z+2})^{1/4} + (\frac{z+2}{z-2}\right)^{1/4}.$$

From this relation we obtain for the monic orthogonal Hermite polynomials $(H_N)_{N\ge 0}$ with respect to the weight $\exp(-x^2/2)$ that

(3.31)

$$\begin{aligned} H_N(z\sqrt{N}) &= (2N)^{N/2}\tilde{H}_N(z/\sqrt{2},(2N)^{-1}) \\ &= 2^{N-1}N^{N/2}U(z)\frac{\exp\left(N(z-\sqrt{z^2-4})^2/8\right)}{(z-\sqrt{z^2-4})^N}\left(1+O\left(\frac{1}{N}\right)\right) \end{aligned}$$

uniformly for z from every compact set contained in $(\mathbb{C}\cup\{\infty\})\setminus[-2,2]$. Passing to the normalized polynomials we see that

(3.32)

$$\begin{aligned} p_N(z\sqrt{N}) &= \frac{1}{(2\pi)^{1/4}(N!)^{1/2}}H_N(z\sqrt{N}) \\ &= \frac{2^N N^{N/2}U(z)}{2(2\pi)^{1/4}(N!)^{1/2}}\frac{\exp\left(N(z-\sqrt{z^2-4})^2/8\right)}{(z-\sqrt{z^2-4})^N}\left(1+O\left(\frac{1}{N}\right)\right), \end{aligned}$$

and, in view of the Stirling formula $N! = \sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}(1+O(1/N))$, we obtain

(3.33)

$$p_N(z\sqrt{N}) = \frac{U(z)\,\exp\left(N(\frac{1}{2}+\log 2+(z-\sqrt{z^2-4})^2/8)\right)}{2(2\pi)^{1/2}(N)^{1/4}\,(z-\sqrt{z^2-4})^N}\left(1+O\left(\frac{1}{N}\right)\right)$$

so that we have

(3.34)
$$p_N(z\sqrt{N}) = \frac{U(z)\,\exp\left(\frac{N}{2}(1+((z-\sqrt{z^2-4})/2)^2)\right)}{2(2\pi)^{1/2}(N)^{1/4}\big((z-\sqrt{z^2-4})/2\big)^N}\left(1+O\left(\frac{1}{N}\right)\right)$$

uniformly in z from any compact subset of $(\mathbb{C}\cup\{\infty\})\setminus[-2,2]$.
For every complex number $z\in\mathbb{C}\setminus[-2,2]$ let $x=x(z)$ be the root of the equation $x+x^{-1}=z$ satisfying $|x|<1$. Then we have

$$x = x(z) = \big(z-\sqrt{z^2-4}\big)/2$$

where $z\mapsto\kappa(z)=\sqrt{z^2-4}$ is, by definition, a univalent analytic function in $\mathbb{C}\setminus[-2,2]$ satisfying the relations $\kappa^2(z)=z^2-4$ and $\kappa(t)>0$ for $t>2$. For every $r\in(0,1)$ the equation $|x(z)|=r$ and the inequality $|x(z)|\le r$ define the ellipse

$$(\Re z)^2/(r^{-1}+r)^2+(\Im z)^2/(r^{-1}-r)^2 = 1 \tag{3.35}$$

with focal points $-2,2$ and the closed exterior domain E_r of this ellipse, respectively. Thus for $z\in E_r$ we have

$$|x(z)| = \big|\big(z-\sqrt{z^2-4}\big)/2\big| \le r < 1,$$

and

$$\Re\big((z-\sqrt{z^2-4})/2\big)^2 \le r^2 < 1. \tag{3.36}$$

Note that for every $r\in(0,1)$

$$\min_{|x(z)|=r}|z| = r^{-1}-r$$

which for $z\in E_r$ implies

$$\begin{aligned}\frac{2}{|z-\sqrt{z^2-4}|} &= \frac{1}{|x(z)|} = |z-x(z)| \le |z|+1 \le (1+|z|^{-1})|z| \\ &\le \left(1+\frac{1}{r^{-1}-r}\right)|z|.\end{aligned} \tag{3.37}$$

Further, for $z\in E_r$ $(0<r<1)$ we can write

$$\begin{aligned}|z-2| &= \big|\big(x(z)-1\big)+\big(x^{-1}(z)-1\big)\big| = 2|x(z)-1||x^{-1}(z)-1| \\ &= 2|x(z)|^{-1}|1-x(z)|^2 \ge 2r^{-1}|1-x(z)|^2 \ge 2r^{-1}|1-r|^2 \\ &= 2(r^{-1/2}-r^{1/2})^2,\end{aligned}$$

and analogously

$$|z+2| \ge 2(r^{-1/2}-r^{1/2})^2.$$

This implies that the function

$$|U(z)| = \left| \left(\frac{z-2}{z+2} \right)^{1/4} + \left(\frac{z+2}{z-2} \right)^{1/4} \right|$$

is bounded above on E_r by a constant depending on $r \in (0,1)$. Applying this estimate along with (3.36) and (3.37) to the relation (3.34) we arrive, for a fixed $r \in (0,1)$ and every $z \in E_r$, at

$$\big| p_N(z\sqrt{N}) \big| \leq C_1(r) N^{-1/4} L^N(r) |z|^N. \tag{3.38}$$

Now we set $z_n = z\sqrt{N/(N-1)}$ and note that, by convexity of $\mathbb{C} \setminus E_r$, $z_N \in E_r$ if $z \in E_r$. Hence, applying (3.38) to p_{N-1}, we see that for every $z \in E_r$

$$\begin{aligned} \big| p_{N-1}\big(z\sqrt{N}\big) \big| = \big| p_{N-1}\big(z_N\sqrt{N-1}\big) \big| &\leq C_1(r)(N-1)^{-1/4} L^{N-1}(r) |z_N|^{N-1} \\ &= C_1(r)\big(1 + 1/(N-1)\big)^{(2N-3)/4} N^{-1/4} L^{N-1}(r) |z|^{N-1}. \end{aligned}$$

Since $\big(1+1/(N-1)\big)^{(2N-3)/4} \to e^{1/2}$ as $N \to \infty$, we conclude that

$$\big| p_{N-1}\big(z\sqrt{N}\big) \big| \leq C_2(r)(N)^{-1/4} L^{N-1}(r) |z|^{N-1}. \tag{3.39}$$

Observe that for every $r \in (0,1)$ the inequality

$$\Im z \geq r^{-1} - r$$

implies, by equation (3.35), that $z \in E_r$. Finally set

$$r = -\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + 1}$$

so that r is the solution of the equation $r^{-1} - r = \delta$ satisfying $0 < r < 1$. We can now conclude from (3.38) and (3.39) that for every $\delta > 0$ with some constants $C(\delta)$ and $M(\delta)$ the inequalities

$$\big| p_N(z\sqrt{N}) \big| \leq C(\delta) N^{-1/4} M^N(\delta) |z|^N \tag{3.40}$$

and

$$\big| p_{N-1}\big(z\sqrt{N}\big) \big| \leq C(\delta)(N)^{-1/4} M^{N-1}(\delta) |z|^{N-1} \tag{3.41}$$

hold for every natural N and every z with $|\Im z| \geq \delta$. We pass now from Hermite polynomials to Hermite functions. Note that for every z satisfying $|\Re z| \geq |\Im z|$ we have

$$|\exp(-z^2/4)| \leq \exp\big((-\Re^2 z + \Im^2 z)/4\big) \leq 1 \tag{3.42}$$

and, therefore,

$$|\varphi_k(z)| \leq |p_k(z)| \tag{3.43}$$

for every $k \in \mathbb{N}$. Applying (3.40) and (3.41), the Lemma follows. $\square$

Corollary 2. *For every real numbers $\delta > 0$ and $A > 0$ there exist such constants $C(\delta, A)$ and $M(\delta)$ that the inequalities*

$$\left|\varphi_N\big((u+t/N)\,d(H)\sqrt{N}\big)\right| \leq C(\delta, A)M^N(\delta)|H|^N \tag{3.44}$$

and

$$\left|\varphi_{N-1}\big((u+t/N)\,d(H)\sqrt{N}\big)\right| \leq C(\delta, A)M^{N-1}(\delta)|H|^{N-1} \tag{3.45}$$

hold for every numbers $u \in [-2,2]$, $t \in [-A, A]$, $N \in \mathbb{N}$ and $H \in \mathbb{R}$ such that $\left|\Im\big((u+t/N)d(H)\big)\right| \geq \delta$.

Proof. Set $v = (u + t/N)\,d(H)$. Since

$$|\Im(d(H))| = \sqrt{\big(\sqrt{1+H^2}-1\big)/2} \leq |H|/2,$$

we have that

$$|\Im(v)| \leq (2 + (A/N))|\Im(d(H))| \leq \big(1 + (A/2N)\big)|H|$$

and

$$|\Im(v)|^N \leq \big(1 + (A/2N)\big)^N |H|^N \leq C(A)|H|^N,$$

since

$$\big(1 + (A/2N)\big)^N \underset{N\to\infty}{\longrightarrow} \exp(A/2).$$

Thus, we proved that

$$|\Im(v)|^N \leq C(A)|H|^N. \tag{3.46}$$

In view of (3.18), $|\Re v| \geq |\Im v|$, and we can apply Lemma 5. Omitting $N^{-1/4}$ in the bound, the proof is completed.

□

Now we will derive upper bounds for $|N^{-1}\tilde{K}_N(\cdot,\cdot,N^{-1})|$ valid for various combinations of the values of its arguments.

Lemma 6. *For every real $\alpha \in (0,1)$ and $A > 0$ there exists a constant $R_1(\alpha, A) > 0$ such that for every real $u \in [-2+\alpha, 2-\alpha]$, $t_1, t_2 \in [-A, A]$, H, and every $N \in \mathbb{N}$ we have*

(3.47)

$$|N^{-1}\tilde{K}_N\big((u+t_1/N)d(H), (u+t_2/N)d(H), N^{-1}\big)| \leq \big(R_1(\alpha, A)\big)^{2N}$$

whenever

$$|\Im(u + t_i/N)d(H)| \leq \beta(\alpha/4) \tag{3.48}$$

holds for $i = 1, 2$.

Proof. Set

$$u_i = u + t_i/N, \quad v_i = (u + t_i/N)d(H) \quad \text{for } i = 1, 2$$

and

$$u = (u_1 + u_2)/2, v = (v_1 + v_2)/2.$$

Let the number $N_0 = N_0(\alpha, A)$ be large enough such that the inequality

$$A/N \leq \alpha/2 \tag{3.49}$$

holds for every $N \geq N_0$. Observe that by (3.49) $u_i \in [-2 + \alpha/2, 2 - \alpha/2], i = 1, 2,$ for $N \geq N_0$. Then it follows from (3.48) and Corollary 1 that $v_1, v_2 \in \overline{S_{\alpha/4,\beta(\alpha/4)}}$. Therefore, $v \in \overline{S_{\alpha/4,\beta(\alpha/4)}}$, too.
First assume now that we have

$$|v_1 - v_2| \leq 1/N. \tag{3.50}$$

Then Remark 1 (with $A = 1$) applies, and it is clear that, for v_1, v_2, $v = (v_1 + v_2)/2 \in \overline{S_{\alpha/4,\beta(\alpha/4)}}$ with v_1, v_2 satisfying (3.50), the function $|\sin\bigl(N\pi(v_1 - v_2)w(v)\bigr)/\bigl(N\pi(v_1 - v_2)\bigr)|$ is bounded above by a constant depending on α and A. Therefore, by Remark 1, this applies to the left hand side of (3.47) as well.
Now, instead of (3.50), we assume that the inequality $|v_1 - v_2| \geq 1/N$ holds. Then, by the Christoffel-Darboux formula (3.9), we have

$$\begin{aligned}
&|N^{-1}\tilde{K}_N\bigl(v_1, v_2, N^{-1}\bigr)| = N^{1/2}|K_N\bigl(\sqrt{N}v_1, \sqrt{N}v_2\bigr)| \\
&\leq \sqrt{N}\bigl(|\varphi_N(\sqrt{N}v_1)||\varphi_{N-1}(\sqrt{N}v_2)| + |\varphi_N(\sqrt{N}v_2)||\varphi_{N-1}(\sqrt{N}v_1)|\bigr).
\end{aligned}$$

Applying Lemma 4 and enlarging, if necessary, the constant in the bound, the Lemma is proved.

□

Lemma 7. *For every $\alpha \in (0, 1)$ and $A > 0$ there exists a constant $R_2(\alpha, A)$ such that for every $u \in [-2 + \alpha, 2 - \alpha]$, $t_1, t_2 \in [-A, A]$, $H \in \mathbb{R}$ and $N \in \mathbb{N}$ we have the relation*

$$|N^{-1}\tilde{K}_N\bigl((u+t_1/N)d(H), (u+t_2/N)d(H), N^{-1}\bigr)| \leq \Gamma(N)\bigl(R_2(\alpha, A)\bigr)^N H^N \tag{3.51}$$

whenever none of the following pairs of inequalities

$$\begin{aligned}
&\bigl|\Im\bigl((u + t_i/N)d(H)\bigr)\bigr| \leq \beta(\alpha/4), \quad i = 1, 2, \\
&\bigl|\Im\bigl((u + t_i/N)d(H)\bigr)\bigr| \geq \beta(\alpha/4)/2, \quad i = 1, 2,
\end{aligned} \tag{3.52}$$

holds.

Proof. Set $u_i = (u + t_i/N), v_i = u_i d(H), i = 1, 2$. Since (3.52) does not hold we have $|\Im(v_i)| < \beta(\alpha/4)/2$ and $|\Im(v_j)| > \beta(\alpha/4)$ for some choice of $i, j \in \{1, 2\}$, $i \neq j$. This implies

$$|v_2 - v_1| \geq \beta(\alpha/4)/2,$$

and by the Christoffel-Darboux formula (3.9) we have

$$\frac{1}{N}\big|\tilde{K}_N\big(v_1, v_2, N^{-1}\big)\big| = \sqrt{N}\big|K_N\big(\sqrt{N}v_1, \sqrt{N}v_2\big)\big|$$
$$\leq \frac{2\sqrt{N}}{\beta(\alpha/4)}\big(|\varphi_N(\sqrt{N}v_1)||\varphi_{N-1}(\sqrt{N}v_2)| + |\varphi_N(\sqrt{N}v_2)||\varphi_{N-1}(\sqrt{N}v_1)|\big).$$

Assume now that, for example, $|\Im(v_1)| < \beta(\alpha/4)/2$ and $|\Im(v_2)| > \beta(\alpha/4)$. In addition we have that $u_1 \in [-2 + \alpha/2, 2 - \alpha/2]$ for $N \geq N(\alpha, A)$. Then, by assertion 2 of Lemma 2, $v_1 \in \overline{S}_{\alpha/4,\beta(\alpha/4)/2} \subseteq \overline{S}_{\alpha/4,\beta(\alpha/4)}$, and we may apply Lemma 4 (with the parameter value $\alpha/4$ instead of α) to bound $\varphi_N(\sqrt{N}v_1)$ and $\varphi_{N-1}(\sqrt{N}v_1)$. Furthermore, we apply Corollary 2 with $\delta = \beta(\alpha/4)$ to bound $\varphi_N(\sqrt{N}v_2)$ and $\varphi_{N-1}(\sqrt{N}v_2)$. Therefore, we arrive at the inequality

$$\frac{1}{N}|\tilde{K}_N\big((u + t_1/N)d(H), (u + t_2/N)d(H), 1/N\big)|$$
$$\leq C(\alpha, A)\sqrt{N}\Gamma(N)M^N(\beta(\alpha/4))H^N.$$

Replacing the quantity $M(\beta(\alpha/4))$ by an appropriate larger constant, we reduce this inequality to the form (3.51). Moreover, this way the bound can be made valid for all $N \in \mathbb{N}$, which proves the Lemma.

□

Lemma 8. *For every $\delta > 0$ and certain constants $C(\delta), M(\delta)$ the inequality*

$$\frac{1}{N}|\tilde{K}_N\big(z_1, z_2, \tfrac{1}{N}\big)| \leq C(\delta)\Gamma(N)R^{2N}(\delta)\,(\Im(z_1))^N(\Im(z_2))^N \tag{3.53}$$

holds for every complex numbers z_1, z_2 such that $|\Im z_i| \geq \delta$ and $|\Re z_i| \geq$ $|\Im z_i|$
$(i = 1, 2)$, and every natural number $N \geq \delta^{-2}$.

Proof. First we consider the case when $\Re z_1 \Re z_2 > 0$. Let

$$f_{a,p}(x) = \frac{1}{\Gamma(p)}a^p x^{p-1}e^{-ax}, p > 0, x > 0,$$

be the *gamma density* [7] with the parameters $a > 0$, $p > 0$, and let $P \geq 0$ and $N \geq 1$ be some integers. Then for every complex number z

satisfying $\Re z \geq |\Im z| > 0$ we have

(3.54)

$$\begin{aligned}
&\int_0^\infty |z+\theta|^{2P} e^{-N\Re(z+\theta)^2/2} d\theta = \int_{\Re z}^\infty |\xi + i\Im z|^{2P} e^{-\frac{N}{2}\Re(\xi+i\Im z)^2} d\xi \\
&\leq \int_{|\Im z|}^\infty |\xi + i\Im z|^{2P} e^{-\frac{N}{2}\Re(\xi+i\Im z)^2} d\xi = \int_{|\Im z|}^\infty (\xi^2 + (\Im z)^2)^P e^{-\frac{N}{2}(\xi^2-(\Im z)^2)} d\xi \\
&= e^{N(\Im z)^2} \int_{|\Im z|}^\infty (\xi^2 + (\Im z)^2)^P e^{-\frac{N}{2}(\xi^2+(\Im z)^2)} d\xi \\
&\leq \frac{2^P\, e^{N(\Im z)^2}}{N^{P+1}|\Im z|} \int_{|\Im z|}^\infty \big(N\big(\xi^2+(\Im z)^2\big)/2\big)^P e^{-\frac{N}{2}(\xi^2+(\Im z)^2)} d\big(N\big(\xi^2+(\Im z)^2\big)/2\big) \\
&= \frac{2^P\, e^{N(\Im z)^2}}{N^{P+1}|\Im z|} \int_{N(\Im z)^2}^\infty \eta^P e^{-\eta} d\eta = \frac{2^P\, e^{N(\Im z)^2}}{N^{P+1}|\Im z|} \Gamma(P+1) \int_{N(\Im z)^2}^\infty f_{1,P}(\eta) d\eta \\
&= \frac{2^P\, e^{N(\Im z)^2}}{N^{P+1}|\Im z|} \Gamma(P+1) e^{-N(\Im z)^2} \left(1 + \frac{N(\Im z)^2}{1!} + \cdots + \frac{[N(\Im z)^2]^P}{P!}\right) \\
&= \frac{2^P \Gamma(P+1)}{N^{P+1}|\Im z|} \left(1 + \frac{N(\Im z)^2}{1!} + \cdots + \frac{[N(\Im z)^2]^P}{P!}\right),
\end{aligned}$$

where we used a well known formula ([7], p. 11) while integrating $f_{1,P}$. If $|\Im z| \geq \delta$ then $N(\Im z)^2 \geq 1$ for $N \geq \delta^{-2}$, and the bound (3.54) yields

$$\int_0^\infty |z+\theta|^{2P} e^{-N\Re(z+\theta)^2/2} d\theta \quad \leq e 2^P N^{-1} \Gamma(P+1) |\Im(z)|^{2P-1} \tag{3.55}$$

whenever $\Re z \geq |\Im z| > 0$. In particular, for $P = N$ and $P = N-1$ we have

$$\int_0^\infty |z+\theta|^{2N} e^{-N\Re(z+\theta)^2/2} d\theta \quad \leq e 2^N N^{-1} \Gamma(N+1) |\Im(z)|^{2N-1} \tag{3.56}$$

and

$$\int_0^\infty |z+\theta|^{2(N-1)} e^{-N\Re(z+\theta)^2/2} d\theta \quad \leq e 2^{N-1} N^{-1} \Gamma(N) |\Im(z)|^{2N-3}. \tag{3.57}$$

Thus, in view of representation (3.12) and the bounds in Lemma 5, for z_1, z_2 such that $|\Im z_i| \geq \delta$ and $\Re z_i \geq |\Im z_i|$ $(i = 1, 2)$ we have

(3.58)
$$\begin{aligned}
&|N^{-1}\tilde{K}_N\big(z_1, z_2, N^{-1})| = |N^{-1/2}K_N\big(\sqrt{N}z_1, \sqrt{N}z_2)| \\
&= (2N)^{-1/2}\bigg|\int_0^\infty \big(\varphi_N(\sqrt{N}z_1+\tau)\varphi_{N-1}(\sqrt{N}z_2+\tau) \\
&\qquad\qquad + \varphi_{N-1}(\sqrt{N}z_1+\tau)\varphi_N(\sqrt{N}z_2+\tau)\big)d\tau\bigg| \\
&= 2^{-1/2}\bigg|\int_0^\infty \big(\varphi_N(\sqrt{N}(z_1+\theta))\varphi_{N-1}(\sqrt{N}(z_2+\theta)) \\
&\qquad\qquad + \varphi_{N-1}(\sqrt{N}(z_1+\theta))\varphi_N(\sqrt{N}(z_2+\theta))\big)d\theta\bigg| \\
&\leq C^2(\delta)N^{-1/2}M^{2N-1}(\delta)\int_0^\infty \big(|z_1+\theta|^N e^{-N\Re(z_1+\theta)^2/4}|z_2+\theta|^{N-1}e^{-N\Re(z_2+\theta)^2/4} \\
&\qquad\qquad + |z_1+\theta|^{N-1}e^{-N\Re(z_1+\theta)^2/4}|z_2+\theta|^N e^{-N\Re(z_2+\theta)^2/4}\big)d\theta \\
&\leq C^2(\delta)N^{-1/2}M^{2N-1}(\delta)\times \\
&\bigg[\bigg(\int_0^\infty |z_1+\theta|^{2N}e^{-N\Re(z_1+\theta)^2/2}d\theta\int_0^\infty |z_2+\theta|^{2(N-1)}e^{-N\Re(z_2+\theta)^2/2}d\theta\bigg)^{1/2} \\
&+\bigg(\int_0^\infty |z_1+\theta|^{2(N-1)}e^{-N\Re(z_1+\theta)^2/2}d\theta\int_0^\infty \big(|z_2+\theta|^{2N}e^{-N\Re(z_2+\theta)^2/2}d\theta\bigg)^{1/2}\bigg].
\end{aligned}$$

Combining this bound with relations (3.56) and (3.57) we find that

(3.59)
$$\frac{|\tilde{K}_N\big(z_1, z_2, N^{-1})|}{N} \leq \frac{e2^{1/2}C^2(\delta)M^{2N-1}(\delta)2^N\Gamma(N)|\Im(z_1)|^N|\Im(z_2)|^N}{\delta^2\,N^{3/2}}.$$

The bound (3.59) applies as well when z_1, z_2 satisfy $|\Im z_i| \geq \delta$ and $\Re z_i \leq -|\Im z|$ $(i = 1, 2)$. Indeed, this follows from (3.11).

Finally, we consider z_1, z_2 such that $|\Re z_i| \geq |\Im z| \geq \delta$ $(i = 1, 2)$ and $\Re z_1 \Re z_2 < 0$. In this case we have

$$\frac{1}{|z_2 - z_1|} \leq \frac{1}{|\Re(z_1)| + |\Re(z_2)|} \leq \frac{1}{|\Im(z_1)| + |\Im(z_2)|} \leq (2\delta)^{-1}$$

and, by (3.9),

$$\begin{aligned}
&\frac{1}{N}|\tilde{K}_N\big(z_1, z_2, N^{-1})| = \frac{1}{\sqrt{N}}|K_N\big(\sqrt{N}z_1, \sqrt{N}z_2)| \\
&\leq \frac{2}{\delta\sqrt{N}}\big(|\varphi_N(\sqrt{N}z_1)||\varphi_{N-1}(\sqrt{N}z_2)| + |\varphi_N(\sqrt{N}z_2)||\varphi_{N-1}(\sqrt{N}z_1)|\big).
\end{aligned}$$

Applying Lemma 5 we obtain

(3.60)

$$|N^{-1}\tilde{K}_N\big(z_1,z_2,N^{-1}\big)| \leq C(\delta)N^{-1}M^{2N-1}(\delta)2^N\Gamma(N)\,(\Im(z_1))^N(\Im(z_2))^N.$$

Setting $R(\delta)=M(\delta)\sqrt{2}$ and comparing 3.59 and 3.60 with the factor N^{-1} omitted, the proof is completed.

$\square$

Corollary 3. *Let $\alpha\in(0,1)$ and A be positive numbers. Then there exist positive constants $C(\alpha,A)$ and $R_2(\alpha)$ such that for every $N\in\mathbb{N}$ and real numbers $u\in[-2,2]$, $t_1,t_2\in[-A,A]$, and H we have*

(3.61)

$$\left|\frac{1}{N}\tilde{K}_N\big(\big(u+\tfrac{t_1}{N}\big)d(H),\big(u+\tfrac{t_2}{N}\big)d(H),\tfrac{1}{N}\big)\right| \leq \Gamma(N)\big(R_3(\alpha,A)\big)^{2N}H^{2N}$$

whenever the inequalities $\big|\Im\big((u+t_i/N)d(H)\big)\big|\geq\ \beta(\alpha/4)/2$, $i=1,2$, hold.

Proof. Set $v_i=(u+t_i/N)\,d(H)$ for $i=1,2$. From (3.18) we conclude that $|\Re v_i|\geq|\Im v_i|$, $i=1,2$. It follows from (3.46) that

$$|\Im(v_i)|^N\leq C(A)|H|^N, i=1,2.$$

Then by Lemma 8 with $\delta=\beta(\alpha/4)/2$ we have

$$\left|\frac{1}{N}\tilde{K}_N\big(\big(u+\tfrac{t_1}{N}\big)d(H),\big(u+\tfrac{t_2}{N}\big)d(H),\tfrac{1}{N}\big)\right|$$
$$\leq C(\beta(\alpha/4)/2,A)\Gamma(N)R^{2N}(\beta(\alpha/4)/2)H^{2N}$$

for every $N\geq\big(\beta(\alpha/4)/2\big)^{-2}$. Substituting R in the latter inequality by a proper larger constant depending also on A, we obtain (3.61) which is valid for all N.

$\square$

Proposition 1. *For every $\alpha\in(0,1)$ and $A>0$ there exists a constant $R(\alpha,A)$ such that for every $u\in[-2+\alpha,2-\alpha]$, $t_1,\ldots,t_n\in[-A,A]$ and $H\in\mathbb{R}$ we have the relations*

(3.62)

$$\left|\frac{1}{N}\tilde{K}_N\big(\big(u+\tfrac{t_1}{N}\big)d(H),\big(u+\tfrac{t_2}{N}\big)d(H),\tfrac{1}{N}\big)\right| \leq \Gamma(N)\big(R(\alpha,A)(1+H^2)\big)^N$$

and

$$\frac{|R_{n,N}^{\mathrm{GUE},\frac{1}{(1+iH)N}}(u+\frac{t_1}{N},\ldots,u+\frac{t_n}{N})|}{N^n} \tag{3.63}$$
$$\leq n!\big(\Gamma(N)\big)^n\big(R(\alpha,A)\big)^{nN}(1+H^2)^{n\left(N+\frac{1}{4}\right)}.$$

Proof. For every $\alpha \in (0,1)$ and $A > 0$, every combination of $t_1 \in [-A, A]$, $t_2 \in [-A, A]$, $H \in \mathbb{R}$, and $N \in \mathbb{N}$ satisfies the assumptions of at least one of the assertions in Lemma 6, Lemma 7 and Corollary 3. Hence, at least one of these upper bounds is valid. Writing $R(\alpha, A) = \max\bigl(\bigl(R_1(\alpha, A)\bigr)^2, R_2(\alpha, A), \bigl(R_3(\alpha, A)\bigr)^2\bigr)$, we obtain

$$\begin{aligned}
&\Bigl|\frac{1}{N}\tilde{K}_N\bigl(\bigl(u+\tfrac{t_1}{N}\bigr)d(H), \bigl(u+\tfrac{t_2}{N}\bigr)d(H), \tfrac{1}{N}\bigr)\Bigr| \\
&\le \max\bigl(\bigl(R_1(\alpha, A)\bigr)^{2N}, \Gamma(N)\bigl(R_2(\alpha, A)\bigr)^N H^N, \Gamma(N)\bigl(R_3(\alpha, A)\bigr)^{2N} H^{2N}\bigr) \\
&\le \bigl(R(\alpha, A)\bigr)^N \max\bigl(1, \Gamma(N)H^N, \Gamma(N)H^{2N}\bigr) \\
&< \bigl(R(\alpha, A)\bigr)^N \max\bigl(1, \Gamma(N)(1+H^2)^{N/2}, \Gamma(N)(1+H^2)^N\bigr) \\
&= \Gamma(N)\bigl(R(\alpha, A)\bigr)^N (1+H^2)^N,
\end{aligned}$$

and the bound (3.62) is proved.

Now we will derive a bound for the correlation function $R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}$. In view of formulas (3.7) and (3.8) we have

$$R_{n,N}^{\mathrm{GUE},\sigma s}(x_1, \ldots, x_n) = \sigma^{-n/2} R_{n,N}^{\mathrm{GUE},s}(\sigma^{-1/2}x_1, \ldots, \sigma^{-1/2}x_n) \tag{3.64}$$

It follows from (2.5) that

$$R_{n,N}^{\mathrm{GUE},\sigma s}(x_1, \ldots, x_n) = \int_0^\infty R_{n,N}^{\mathrm{HSE},u/N^2} \gamma_{N^2,\sigma s}(u) du, \tag{3.65}$$

where

$$\gamma_{m,s}(u) = \begin{cases} (2^{m/2}s^{-m/2}\Gamma(m/2))^{-1}u^{(m/2)-1}\exp\left(-u/(2s)\right), \text{if}\, u \ge 0, \\ 0, \text{if}\, u < 0. \end{cases} \tag{3.66}$$

Since $(x_1, \ldots, x_n) \mapsto R_{n,N}^{\mathrm{GUE},s}(x_1, \ldots, x_n)$ is an entire function, formula (3.64) can be used to analytically continue the function $R_{n,N}^{\mathrm{GUE},\sigma s}(x_1, \ldots, x_n)$ in the parameter σ to the domain $\Re\sigma > 0$. On the other hand, an analytic continuation can be obtained from formulas (3.65) and (3.66) (we omit the check of this fact requiring routine bounds for $\gamma_{m,s}$ in the parameter $s \in \mathbb{C}$). Since these continuations obviously agree, both (3.64) and (3.65) are valid for complex s with $\Re s > 0$. In particular, we have the identity

$$R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}(x_1, \ldots, x_n) = d^n(H) R_{n,N}^{\mathrm{GUE},1/N}(x_1 d(H), \ldots, x_n d(H)). \tag{3.67}$$

Along with the determinantal representation (3.7) (which also admits an analytic continuation) and inequality (3.62), this leads to

(3.68)

$$\begin{aligned}
&N^{-n}|R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}(u+t_1/N,\dots,u+t_n/N)| \\
&= N^{-n}|d(H)|^n R_{n,N}^{\mathrm{GUE},1/N)}((u+t_1/N)d(H),\dots,(u+t_1/N)d(H))| \\
&\le n!|d(H)|^n \max\nolimits_{0\le i<j\le n} |N^{-1}\tilde{K}((u+t_i/N)d(H),(u+t_j/N)d(H))|^n \\
&\le n!|d(H)|^n\big(\Gamma(N)\big(R(\alpha,A)\big)^N(1+H^2)^N\big)^n \\
&= n!\big(\Gamma(N)\big)^n\big(R(\alpha,A)\big)^{nN}(1+H^2)^{n(N+1/4)}
\end{aligned}$$

since $|d(H)|=(1+H^2)^{1/4}$ by (3.14).

$\square$

Proof of Theorem. As explained in the end of Section 2 we need to show that

(3.69)

$$\begin{aligned}
&\frac{1}{(Nw(u))^n}\int_{-\infty}^{\infty}\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1-ih\sqrt{2}/N)N)}\Big(u+\frac{t_1}{Nw(u)},\dots,u+\frac{t_n}{Nw(u)}\Big)dh \\
&\qquad\qquad\underset{N\to\infty}{\longrightarrow}\frac{1}{\sqrt{2\pi}}\det\Big(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Big)_{i,j=1}^n
\end{aligned}$$

uniformly in $u,t_1,\dots,t_n$ subject to the conditions formulated in the the statement of the Theorem. Throughout the proof we will assume that these conditions are satisfied and omit the arguments of $R_{n,N}^{\mathrm{GUE},\cdot}$ whenever that possible. The existence of the integral in the left hand side of relation (3.69) will be a consequence of our estimates.

Note that $R_{n,N}^{\mathrm{GUE},s}(-u_1,\dots,-u_n) = R_{n,N}^{\mathrm{GUE},s}(u_1,\dots,u_n)$ since $K_N(-u_1,-u_2)=K_N(u_1,u_2)$ by (3.11). In view of this property it suffices to prove (3.69) for $u\ge 0$ only. Observe that for real h

$$\phi_{N^2}(-h)=\overline{\phi_{N^2}(h)}$$

and

$$R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)}(u_1,\dots,u_n)=\overline{R_{n,N}^{\mathrm{GUE},1/((1-ih\sqrt{2}/N)N)}}(u_1,\dots,u_n)$$

for $u_1,\dots,u_n\in\mathbb{R}$ which shows that (3.69) is established if we prove

$$\begin{aligned}
&\frac{1}{(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)}\big)dh \\
&\qquad\underset{N\to\infty}{\longrightarrow}\frac{1}{2\sqrt{2\pi}}\det\Big(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\Big)_{i,j=1}^n.
\end{aligned}\tag{3.70}$$

It follows from the central limit theorem for densities that

$$\int_0^\infty \Re\big(\phi_{N^2}(h)\big)dh \underset{N\to\infty}{\longrightarrow} \frac{1}{2\sqrt{2\pi}}. \tag{3.71}$$

Here $\phi_{N^2}(\cdot)$ is a prelimiting characteristic function and $1/2\sqrt{2\pi}$ is half the value of the limiting standard Gaussian density at 0. Thus, to prove (3.70) it suffices to establish the following relation

$$\begin{aligned}&\frac{1}{(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)}\big)dh\\ &\qquad\qquad - \det\bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\bigg)_{i,j=1}^n\int_0^\infty \Re\big(\phi_{N^2}(h)\big)dh\\ &=: I(u,N)-J(N) \underset{N\to\infty}{\longrightarrow} 0\end{aligned} \tag{3.72}$$

under the same uniformity constraints as above. Omitting the arguments $t_1,\dots,t_n$, we set

$$S_n(u,H)=\det\bigg(\frac{\sin\big(\pi(t_i-t_j)d(H)w(ud(H))/w(u)\big)}{\pi(t_i-t_j)d(H)}\bigg)_{i,j=1}^n$$

and

$$S_n=S_n(u,0)=\det\bigg(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\bigg)_{i,j=1}^n.$$

For a given $\alpha\in(0,1)$, let $\overline{H}(\alpha)>0$ be a number whose existence is guaranteed by Lemma 3. Let $\epsilon>0$ be an arbitrary positive number, and $H_0=H_0(\alpha,A,\epsilon)\in(0,\overline{H}(\alpha))$ be small enough to ensure that

$$|S_n(u,H)-S_n|\le \epsilon 2\sqrt{2}/\pi \tag{3.73}$$

for all $u\in[-2+\alpha,2-\alpha]$, $|H|\le H_0$ and $t_1,\dots,t_n$ satisfying $|t_1|\le A,\dots,|t_n|\le A$. Such a number exists because of continuity of the sin-kernel and continuity near 0 of the function $d(\cdot)$. Performing the change of variable $H=h\sqrt{2}/N$, we may write

(3.74)

$$\begin{aligned}I(u,N)=&\frac{1}{(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(h)R_{n,N}^{\mathrm{GUE},1/((1+ih\sqrt{2}/N)N)}\big)dh\\ =&\frac{N}{\sqrt{2}(Nw(u))^n}\int_0^\infty \Re\big(\phi_{N^2}(NH/\sqrt{2})R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}\big)dH\\ =&\frac{N}{\sqrt{2}(Nw(u))^n}\int_0^{H_0}\dots\,dH+\frac{N}{\sqrt{2}(Nw(u))^n}\int_{H_0}^\infty\dots\,dH\\ =:&\, I_1(\epsilon,u,N)+I_2(\epsilon,u,N).\end{aligned}$$

Quite analogously, we have

(3.75)

$$\begin{aligned}
J(N) =& S_n \int_0^\infty \Re\big(\phi_{N^2}(h)\big)dh = \frac{N\,S_n}{\sqrt{2}} \int_0^\infty \Re\big(\phi_{N^2}(NH/\sqrt{2})\big)dH \\
=& \frac{N\,S_n}{\sqrt{2}} \int_0^{H_0} \Re\big(\phi_{N^2}(NH/\sqrt{2})\big)dH + \frac{N\,S_n}{\sqrt{2}} \int_{H_0}^\infty \Re\big(\phi_{N^2}(NH/\sqrt{2})\big)dH \\
=& :\ J_1(\epsilon, N) + J_2(\epsilon, N).
\end{aligned}$$

With this notation we need to show that

$$I_1(\epsilon, u, N) - J_1(\epsilon, N) \underset{N\to\infty}{\longrightarrow} 0, \tag{3.76}$$

$$I_2(\epsilon, u, N) \underset{N\to\infty}{\longrightarrow} 0, \tag{3.77}$$

and

$$J_2(\epsilon, N) \underset{N\to\infty}{\longrightarrow} 0 \tag{3.78}$$

uniformly in $u \in [-2+\alpha, 2-\alpha]$ and $t_i \in [-A, A], i = 1, \ldots, n$.

By Proposition 1, for every ϵ, we obtain

(3.79)

$$\begin{aligned}
&|I_2(\epsilon, u, N)| \\
&\leq \frac{N}{\sqrt{2}(w(u))^n} n!\big(\Gamma(N)\big)^n \big(R(\alpha, A)\big)^{nN} \int_{H_0}^\infty (1+H^2)^{n(N+1/4)} \big|\phi_{N^2}(\tfrac{NH}{\sqrt{2}})\big| dH.
\end{aligned}$$

Since

$$|\phi_{N^2}(NH/\sqrt{2})| = (1+H^2)^{-(N^2/4)}$$

and

(3.80)

$$\int_L^\infty (1+H^2)^{-K} dH < \frac{1}{(1+L^2)^{K-1}} \int_0^\infty (1+H^2) dH = \frac{\pi}{2(1+L^2)^{K-1}}$$

for every $K > 1$, we have, in particular, that

$$\int_L^\infty |\phi_{N^2}(NH/\sqrt{2})| dH < \frac{\pi}{2(1+L^2)^{N^2/4-1}}. \tag{3.81}$$

Note that $N^2/4 - nN - 1/4 > 1$ whenever $N > N_0 = 2n + \sqrt{4n^2 + 5/4}$. For every $N > N_0$ we obtain from (3.80) with $K = N^2/4 - nN - 1/4$ and $L = H_0$ that

(3.82)

$$I_2(\epsilon, u, N) \leq \frac{\pi N}{2\sqrt{2}(w(u))^n} n!\big(\Gamma(N)\big)^n \big(R(\alpha, A)\big)^{nN} \frac{1}{\big(1+H_0^2\big)^{N^2/4 - nN - 5/4}}.$$

Since

$$\inf_{u\in[-2+\epsilon,2-\epsilon]} |w(u)| = w(2-\epsilon) \tag{3.83}$$

and, by the Stirling formula, $(1+H_0^2)^{N^2/4}$ tends to infinity more rapidly than any fixed power of $\Gamma(N)$ (or, moreover, the $N-$th power of any positive number), we conclude from (3.82) that

$$I_2(\epsilon,u,N) \underset{N\to\infty}{\longrightarrow} 0 \tag{3.84}$$

uniformly in $u\in[-2+\alpha,2-\alpha]$ and $t_i\in[-A,A], i=1,\ldots,n$.
The check of (3.78) is even more straightforward. Indeed, using (3.80) with $K=N^2/4$ and $L=H_0$, we have

$$\begin{aligned}
&|J_2(\epsilon,N)| \\
&=\frac{N\,S_n}{\sqrt{2}}\left|\int_{H_0}^{\infty}\Re\big(\phi_{N^2}(NH/\sqrt{2})\big)dH\right| \le \frac{N\,S_n}{\sqrt{2}}\int_{H_0}^{\infty}\big|\phi_{N^2}(NH/\sqrt{2})\big|dH \\
&\quad=\frac{N\,S_n}{\sqrt{2}}\int_{H_0}^{\infty}(1+H^2)^{-(N^2/4)}dH=\frac{\pi N\,S_n}{2\sqrt{2}(1+L^2)^{N^2/4-1}},
\end{aligned}$$

and (3.78) follows.
Now we complete the proof by establishing (3.76). Recall that we omit the arguments of $R_{n,N}^{\mathrm{GUE},1/N}$. All bounds and limit transitions are uniform with respect to $u\in[-2+\alpha,2-\alpha]$ and $t_1,t_2,\ldots,t_n$ satisfying $|t_1|\le A,\ldots,|t_n|\le A$. We have

(3.85)

$$\begin{aligned}
&|I_1(\epsilon,u,N)-J_1(\epsilon,N)| \\
&\le\frac{N}{\sqrt{2}}\int_0^{H_0}\left|\left(\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n}-S_n\right)\phi_{N^2}(NH/\sqrt{2})\right|dH \\
&\le\frac{N}{\sqrt{2}}\int_0^{H_0}\left|\left(\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n}-S_n(u,H)\right)\phi_{N^2}(NH/\sqrt{2})\right|dH \\
&\qquad+\frac{N}{\sqrt{2}}\int_0^{H_0}\left|(S_n(u,H)-S_n)\,\phi_{N^2}(NH/\sqrt{2})\right|dH \\
&=: D_1^{(1)}(\epsilon,u,N)+D_2^{(2)}(\epsilon,u,N).
\end{aligned}$$

It follows from the determinantal formula (3.7) and Lemma 3 that

$$\sup_{0\le H\le H_0}\left|\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n}-S_n(u,H)\right| \underset{N\to\infty}{\longrightarrow} 0 \tag{3.86}$$

uniformly in $u\in[-2+\alpha,2-\alpha]$, and $|t_i|\le A\,(i=1,\ldots,n)$.

Therefore, using (3.81), we obtain

$$D_1^{(1)}(\epsilon,u,N) \leq \sup_{0\leq H\leq H_0}\left|\frac{R_{n,N}^{\mathrm{GUE},1/((1+iH)N)}}{(Nw(u))^n} - S_n(u,H)\right| \int_0^\infty \left|\phi_{N^2}(h)\right| dh \underset{N\to\infty}{\longrightarrow} 0$$

since

$$\int_0^\infty \left|\phi_{N^2}(h)\right| dh \underset{N\to\infty}{\longrightarrow} \frac{1}{2\sqrt{2\pi}}. \tag{3.87}$$

Finally, we see from (3.73) and (3.87) that

$$D_1^{(2)}(\epsilon,u,N) \leq \epsilon.$$

Since ϵ is arbitrary small, this completes the proof. □

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